Connected but Path Discrete

Michael Bradley

August 14, 2021

Abstract

We construct a topological space with the order topology which is connected but path discrete. In other words, there is no path between any pair of distinct points. This space provides a helpful illustration of why connectivity does not imply any amount of path connectivity.

Definition 1 (Path Discrete)**.** A topological space is path discrete if there does not exist a path between any pair of distinct points.

Definition 2 (Linear Continuum)**.** An ordered set X is a linear continuum if

1. X is densely ordered. In other words,

$$\forall x\ y \in X, (x < y \implies \exists a \in X, x < a < y)$$

2. X has the least upper bound property.

Theorem 1. If a topological space has the order topology and is a linear continuum, then it is connected. See Theorem 24.1 from *Topology* by James Munkres.

Theorem 2 (Intermediate Value Theorem)**.** See Theorem 24.3 from *Topology* by James Munkres.

Theorem 3. Every nonempty open subset of $\mathbb{R}$ contains an element from $\mathbb{Q}$.

Proof. Note that every open subset of $\mathbb{R}$ is precisely a union of open intervals. □

Definition 3. The topological space Ω is defined as

$$\Omega = [0,1]^\omega = [0,1] \times [0,1] \times \cdots$$

with the dictionary order topology. In other words, Ω consists of every sequence $(x_i)_{i=1}^\infty$ of real numbers $x_i \in [0,1]$.

Theorem 4. Ω is connected.

Proof. By Theorem 1, it suffices to show Ω is a linear continuum.

We must show Ω is densely ordered. Let $x, y \in \Omega$ such that $x < y$. By definition, $\exists n \in \mathbb{N}$ such that $x_n < y_n$ and $\forall m < n$, we have $x_m = y_m$. Let $u_i = x_i$ for all $i < n$. Let $u_n = (x_n + y_n)/2 \in [0, 1]$. Let $u_i = 0$ for all $i > n$. Then, $u \in \Omega$ and $x < u < y$. Thus, Ω is densely ordered.

We must show Ω has the least upper bound property. Let $U_0 \subset \Omega$. For $i \geq 1$, let

$$u_i = \begin{cases} \sup\limits_{x \in U_{i-1}} x_i & U_{i-1} \neq \varnothing \\ 0 \end{cases} \qquad \in [0, 1]$$

$$U_i = \{\, x \in U_{i-1} \mid x_i = u_i \,\}$$

Suppose $\exists z \in \Omega$ such that z is an upper bound of U_0 and $z < u$. Then, $\exists n \in \mathbb{N}$ such that $z_n < u_n$ and $\forall m < n$, we have $z_m = u_m$. Then, since $z_n \geq 0$,

$$z_n < \sup_{x \in U_{n-1}} x_n$$

Thus, $\exists x \in U_{n-1} \subset U_0$ such that $z_n < x_n$. However, $\forall m < n$, we have $z_n = u_n = x_n$. Hence, $z < x$ and z is not an upper bound for U_0 This is a contradiction. Therefore, u is the least upper bound of U_0. □

Theorem 5. Ω is path discrete.

Proof. Let $x, y \in \Omega$ such that $x < y$. Suppose there exists a continuous function $f : [0, 1] \to \Omega$ such that $f(0) = x$ and $f(1) = y$. We must demonstrate a contradiction.

Since Ω is densely ordered, choose $u \in \Omega$ such that $x < u < y$. Then, $\exists k, h \in \mathbb{N}$ such that $x_k < u_k$ and $u_h < y_h$. Let $n = \max(k, h) + 1$. For each $r \in [0, 1]$, let

$$\alpha_{ri} = \begin{cases} u_i & i < n \\ r & i = n \\ 0 & i > n \end{cases} \qquad \beta_{ri} = \begin{cases} u_i & i < n \\ r & i = n \\ 1 & i > n \end{cases} \qquad U_r = (\alpha_r, \beta_r)$$

Note, $\alpha_r, \beta_r \in \Omega$. Again, since Ω is densely ordered, $\alpha_r < \beta_r$ implies $U_r \neq \varnothing$, and hence $f^{-1}(U_r) \neq \varnothing$ by Theorem 2. Since f is continuous, $f^{-1}(U_r)$ is open, and thus $\exists q_r \in \mathbb{Q}$ such that $q_r \in f^{-1}(U_r)$ by Theorem 3.

Suppose $q_a = q_b$. If $a < b$, then $\beta_a < \alpha_b$, which implies $U_a \cap U_b = \varnothing$. However, $U_a \ni q_a = q_b \in U_b$. Hence, $a = b$. Thus, q provides an injection from the real interval $[0, 1]$ into $\mathbb{Q}$. This is a contradiction. □